Error-Tolerating Bell Inequalities via Graph States

Qing Chen,[1,2] Sixia Yu,[1,2] and C. H. Oh[2]

[1]Hefei National Laboratory for Physical Sciences at Microscale and Department of Modern Physics, University of Science and Technology of China, Hefei 230026, P.R. China

[2]Centre for Quantum Technologies and Physics Department, National University of Singapore, 2 Science Drive 3, Singapore 117542

(Dated: March 19, 2009)

We investigate the Bell inequalities derived from the graph states with violations detectable even with the presence of noises, which generalizes the idea of error-correcting Bell inequalities [Phys. Rev. Lett. **101**, 080501 (2008)]. Firstly we construct a family of valid Bell inequalities tolerating arbitrary t-qubit errors involving $3(t+1)$ qubits, e.g., 6 qubits suffice to tolerate single qubit errors. Secondly we construct also a single-error-tolerating Bell inequality with a violation that increases exponentially with the number of qubits. Exhaustive computer search for optimal error-tolerating Bell inequalities based on graph states on no more than 10 qubits shows that our constructions are optimal for single- and double-error tolerance.

PACS numbers: 03.67.Pp, 03.65.Ud, 03.67.Mn

Quantum theory is inconsistent with local hidden variable (LHV) theory, which is quantitatively characterized by the violations of Bell inequalities [1]. Experimentally the violations have been confirmed only up to a certain extent [2, 3]. To close the loopholes many efforts have been devoted to designing Bell inequalities involving multi observers and multi measurement settings in search for larger violations [4, 5, 6, 7]. Based on the graph state [8], which is an essential resource in the one-way computing [9], the multi-observer Bell inequalities are extensively studied [6, 7, 10, 11, 12].

In general the Bell inequality is designed for some special multipartite entangled quantum states, which may undergo some inevitable errors. A powerful approach to fight the errors is to use the quantum error-correcting codes. To protect our systems from single qubit errors the simplest Bell inequality involves 10 qubits by using the perfect 5-qubit code. In this case active decodings at the detection steps are required. Recently an error-correcting Bell inequality has been proposed based on some codewords of quantum error-correction codes in which only passive detections are required [13]. To tolerate single qubit errors a minimum number of 11 qubits are involved in a valid Bell inequality. Here we shall employ the term *error-tolerating* instead of the original term *error-correcting* because the violation can be detected without involving any encoding-decoding procedures even when there are some errors happened to the quantum state.

On the other hand the graph state turns out to be a systematic tool for constructing good quantum codes, either additive or nonadditive, binary or nonbinary [14, 15, 16, 17, 18, 19]. It is therefore of much interest to combine those two ideas: the Bell inequalities from graph states and error tolerating to gain some new insights. In this letter we firstly prove that every graph state can be used to build an error-tolerating Bell operator, and then by using some special graph states we build a valid Bell inequality (with violations) on $3(t+1)$ qubits to tolerate up to t-qubit errors. As a result, only 6 qubits are required instead of the original 11 qubits [13] to tolerate single qubits errors so that our error-tolerating Bell inequality can be possibly tested under current experimental conditions [20, 21]. Also we have constructed a single-error tolerating Bell inequality with a violation that increases exponentially with the number of the qubits.

A graph $G=(V,E)$ is composed of a set V of n vertices and a set of edges $E\subset V\times V$, i.e., two different vertices $a,b\in V$ are connected iff $(a,b)\in E$. The neighborhood of a vertex a is defined to be the set of all the vertices that are connected to a, i.e., $N_a=\{b\in V|(a,b)\in E\}$. The graph state $|G\rangle$ corresponding to a graph G on n vertices is an n-qubit state that is the unique joint +1 eigenstate of the following n commuting observables

$$\mathcal{G}_a=\mathcal{X}_a\prod_{b\in N_a}\mathcal{Z}_b:=\mathcal{X}_a\mathcal{Z}_{N_a},\quad a\in V, \tag{1}$$

which are referred to as *vertex-stabilizers*, i.e., $\mathcal{G}_a|G\rangle=|G\rangle$, for $a=1,...,n$. Here $\mathcal{X}_a,\mathcal{Y}_a,\mathcal{Z}_a$ are three Pauli operators acting on the qubit a, and furthermore an operator subscripted by a subset stands for the product of the same operator indexed by all the qubits in the subset. For an arbitrary vertex subset $\omega\subseteq V$ the observable $\mathcal{G}_\omega=\prod_{a\in\omega}\mathcal{G}_a$ also stabilizes the graph state.

One distinct advantage of the graph state is that any Pauli operator is equivalent to the product of a phase flip operator and a stabilizer of the given graph state. In fact from Eq. (1) it follows that $\mathcal{Z}_\delta\mathcal{X}_\omega\propto\mathcal{Z}_{\delta\triangle N_\omega}\mathcal{G}_\omega$ for arbitrary $\omega,\delta\subseteq V$, where $N_\omega=\triangle_{v\in\omega}N_v$ is the neighborhood of a subset ω with $C\triangle D:=C\cup D-C\cap D$ being the *symmetric difference* of any two sets C,D. Thus all the nondegenerate Pauli operators acting on no more than t qubits will be equivalent to some phase flips $\mathcal{Z}_C$ when acting on the graph state with $C\in\mathbb{C}_t$ where

$$\mathbb{C}_t=\left\{\delta\triangle N_\omega\,\Big|\,|\omega\cup\delta|\le t\right\} \tag{2}$$

is referred to as the t-coverable set. In general we have $\mathbb{C}_0=\{\emptyset\}$. Based on the t-coverable set a graphical approach has been developed to construct the quantum error correction codes [16, 17, 18].

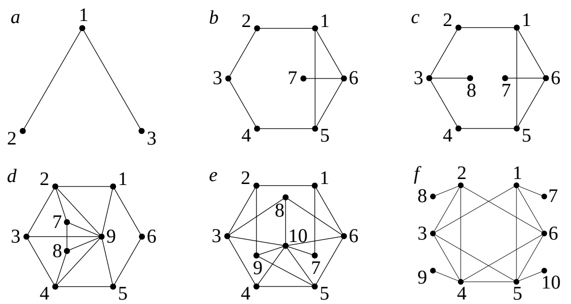

FIG. 1: Some selected optimal graphs whose Bell operators $\mathcal{B}_t(G)$ have the largest violations: Figs.1a, 1$b \sim 1e$ for $t = 0$ and Fig.1f for $t = 1$.

For any given graph G and the corresponding graph state $|G\rangle$, since $\{\mathcal{Z}_C \mid C \in \mathbb{C}_t\}$ is exactly the set of all the representative nondegenerate errors acting nontrivially on no more than t qubits, we introduce the t-error-tolerating Bell operator in a similar manner as in Ref. [13]

$$\mathcal{B}_t(G) = \sum_{C\in\mathbb{C}_t} \mathcal{Z}_C|G\rangle\langle G|\mathcal{Z}_C^\dagger. \tag{3}$$

It is obvious that $\mathcal{B}_0(G)$ is exactly the Bell operator for the 3-setting Bell inequality constructed from the graph state [6]. For a simple example, we consider the 3-qubit GHZ state corresponding to the star graph $\wedge$ as shown in Fig 1a. We have $\mathbb{C}_t(\wedge) = 2^V$ for $t \geq 1$ and as a result

$$\begin{aligned} 8\mathcal{B}_0(\wedge) = \mathbb{1}_1\mathbb{1}_2\mathbb{1}_3 + \mathcal{X}_1\mathcal{Z}_2\mathcal{Z}_3 + \mathcal{Z}_1\mathcal{X}_2\mathbb{1}_3 + \mathcal{Z}_1\mathbb{1}_2\mathcal{X}_3 \\ +\mathcal{Y}_1\mathcal{Y}_2\mathcal{Z}_3 + \mathcal{Y}_1\mathcal{Z}_2\mathcal{Y}_3 + \mathbb{1}_1\mathcal{X}_2\mathcal{X}_3 - \mathcal{X}_1\mathcal{Y}_2\mathcal{Y}_3, \end{aligned} \tag{4}$$

while $8\mathcal{B}_t(\wedge) = \mathbb{1}$ $(t = 1, 2, 3)$. The following proposition ensures that the expectation values of the Bell operator $\mathcal{B}_t(G)$ under the corresponding graph state is error-tolerating.

Proposition 1: For a given graph G on n vertices with the corresponding n-qubit graph state $|G\rangle$ and an arbitrary trace-preserving completely positive map $\mathcal{E}_t$ described by the Kraus operators $\{E_i\}_{i\in I}$ that are linear combinations of Pauli operators nontrivially acting on at most t qubits, we have

$$\mathrm{Tr}\Big(\mathcal{B}_t(G)\mathcal{E}_t(|G\rangle\langle G|)\Big) = 1. \tag{5}$$

Proof. An arbitrary Pauli operator acting nontrivially on no more than t qubits can be generally written as $\mathcal{X}_\omega\mathcal{Z}_\delta$ with $|\delta \cup \omega| \leq t$ and, when acting on a graph state $|G\rangle$, is proportional to a phase flip $\mathcal{Z}_C$ up to a phase factor with $C = \delta \triangle N_\omega \in \mathbb{C}_t(G)$. As a result we have expansion

$$E_i = \sum_{C\in\mathbb{C}_t} \mathcal{Z}_C \sum_{\omega\in\Omega_C} \lambda^i_{C,\omega}\mathcal{G}_\omega, \tag{6}$$

where $\Omega_C = \{\omega \subseteq V \mid \exists\delta \subseteq V, |\delta\cup\omega| \leq t\ s.t.\ \delta \triangle N_\omega = C\}$. From the trace-preserving condition for $\mathcal{E}_t$, i.e., $\sum_{i\in I} E_i^\dagger E_i = \mathbb{1}$, it follows that

$$\sum_{i\in I}\sum_{C\in\mathbb{C}_t}\sum_{\omega,\omega'\in\Omega_C} (\lambda^i_{C,\omega'})^*\mathcal{G}_{\omega'}\lambda^i_{C,\omega}\mathcal{G}_\omega = \mathbb{1} \tag{7}$$

which leads to

$$\sum_{i\in I}\sum_{C\in\mathbb{C}_t}\left|\sum_{\omega\in\Omega_C}\lambda^i_{C,\omega}\right|^2 = 1 \tag{8}$$

when averaged in the graph state $|G\rangle$. As an immediate consequence

$$\begin{aligned} \mathrm{Tr}\Big(\mathcal{B}_t(G)\mathcal{E}_t(|G\rangle\langle G|)\Big) &= \sum_{i\in I}\langle G|E_i^\dagger\mathcal{B}_t(G)E_i|G\rangle \\ &= \sum_{i\in I, C\in\mathbb{C}_t} |\langle G|\mathcal{Z}_C E_i|G\rangle|^2 \\ &= \sum_{i\in I, C\in\mathbb{C}_t}\left|\sum_{\omega\in\Omega_C}\lambda^i_{C,\omega}\right|^2, \end{aligned} \tag{9}$$

which yields the desired result. □

The above proposition shows that the expectation value of the Bell operator $\mathcal{B}_t(G)$ is the same when measured in the graph state $|G\rangle$ no matter whether there are some errors acting on up to t qubits or not. That is why the Bell operator Eq. (3) is referred to as error-tolerating. Next we shall investigate the maximal value of the Bell operator $\mathcal{B}_t(G)$ in the local hidden variable models in order to have valid error-tolerating Bell inequalities.

For a given graph G on n vertices with the corresponding n-qubit graph state $|G\rangle$, the Bell operator can be equivalently rewritten as

$$\mathcal{B}_t(G) = \frac{1}{2^n}\sum_{S\subseteq V}\sum_{C\in\mathbb{C}_t}(-1)^{|C\cap S|}\mathcal{G}_S. \tag{10}$$

Now we have n observers with each observer having 3 measurement settings corresponding to 3 Pauli operators $\mathcal{X}_a, \mathcal{Y}_a$, and $\mathcal{Z}_a$ and they can assume values $x_a, y_a, z_a = \pm 1$ independently. By exhaustively calculating all possible realistic assignments, we can determine the LHV bound

$$\mathcal{D}_t(G) = \max_{\mathrm{LHV}}\langle\mathcal{B}_t(G)\rangle_c. \tag{11}$$

If for a given graph G we have $\mathcal{D}_t(G) < 1$, then we have a valid Bell inequality, i.e., it can be violated. In this case the Bell inequality is error-tolerant because even there are up to t arbitrary qubit errors, if one measures the Bell operator $\mathcal{B}_t(G)$ in the corresponding graph state the violation can still be measured. The smaller the $\mathcal{D}_t(G)$ the larger the violation of the Bell inequality.

Since local clifford (LC) operations are special permutations $\{\mathcal{X}_i, \mathcal{Y}_i, \mathcal{Z}_i\} \to \pm\{\mathcal{X}_i, \mathcal{Y}_i, \mathcal{Z}_i\}$, the LHV value $\mathcal{D}_t(G)$ for the Bell operators defined on LC-equivalent graph states [8, 15] should be the same. For example, the complete graph and the star graph are LC-equivalent, which correspond to the GHZ state, and therefore their LHV bounds are equal. Starting from the GHZ state valid Bell inequalities have been established when there is no error at all [5, 6, 13], i.e., for complete graph K_n we have $\mathcal{D}_0(K_n) < 1$ for $n > 2$. However if some

errors are permitted there is no valid Bell inequality that can be built from the Bell operator defined on K_n since we have

Proposition 2: $\mathcal{D}_t(K_n) = 1$ for $t \geq 1$.

Proof. Given a complete graph K_n, it is easy to obtain the t-coverable set for $t \geq 1$ as

$$\mathbb{C}_t = \left\{ C \subseteq V \middle| |C| \leq t \text{ or } |C| \geq n - t \right\}. \tag{12}$$

It is clear that $C \in \mathbb{C}_t$ iff $V - C \in \mathbb{C}_t$ and because $(-1)^{|C \cap S|} + (-1)^{|(V-C)\cap S|}$ is zero when $|S|$ is odd we have in general $\sum_{C \in \mathbb{C}_t} (-1)^{|C \cap S|} = 0$ if $|S|$ is odd. When $|S|$ is an even number we have $\mathcal{G}_S = \mathcal{Y}_S$ for the complete graph (note that $\mathcal{G}_\emptyset = \mathcal{Y}_\emptyset = \mathbb{1}$). For a given assignment of LHV values to $\mathcal{Y}_S$ let $A \subseteq V$ be the subset of qubits on which $\mathcal{Y}_a$ is assigned to value -1 for all $a \in A$. Thus $\mathcal{Y}_S$ is assigned to the value $(-1)^{|S \cap A|}$ so that the LHV value for the Bell operator reads

$$\langle \mathcal{B}_t(K_n) \rangle = \frac{1}{2^n} \sum_{S \subseteq V} \sum_{C \in \mathbb{C}_t} (-1)^{|S \cap (C \triangle A)|}, \tag{13}$$

which equals to 1 if $A \in \mathbb{C}_t$ and 0 otherwise. $\square$

Instead of a single copy of the star graph we consider now two or more copies of the star graph whose graph state is a direct product of some GHZ states. Let $\wedge^2 = \wedge_1 \oplus \wedge_2$ be the graph on 6 vertices $V_1 \cup V_2$ that is composed of two copies of the star graph $\wedge$ on 3 vertices whose vertex sets are denoted as V_1 and V_2. Its 1-coverable set is obviously $\{C | C \subseteq V_1 \text{ or } C \subseteq V_2\}$ from which the 1-error-tolerating Bell operator can be calculated

$$\mathcal{B}_1(\wedge^2) = \mathbb{1} \otimes \mathcal{B}_0(\wedge_2) + \mathcal{B}_0(\wedge_1) \otimes (\mathbb{1} - \mathcal{B}_0(\wedge_2)). \tag{14}$$

It is easy to see that its LHV value $\langle \mathcal{B}_1(\wedge^2) \rangle_c = b_2 + b_1(1 - b_2)$ with $b_i = \langle \mathcal{B}_0(\wedge_i) \rangle_c \in \{-1/4, 1/4, 3/4\}$ for $i = 1, 2$ reaches its maximum when $b_1 = b_2 = 3/4$. Therefore the LHV bound is $\mathcal{D}_1(\wedge^2) = 15/16 < 1$, which means that we have a valid Bell inequality using 6 qubits instead of 11 qubits in [13] to tolerate single-qubit errors.

As a direct generalization we consider the graph $\wedge^m = \oplus_{i=1}^m \wedge_i$ on $3m$ vertices $\cup_{i=1}^m V_i$ that composes of m copies of the star graph $\wedge$ on 3 vertices whose vertices sets are V_i. The 1-coverable set for this graph can be easily found to be $\{C | C \subseteq V_i \text{ for some } i\}$ so that the 1-error-tolerating Bell operator can be recursively expressed by

$$\begin{aligned} \mathcal{B}_1(\wedge^{m+1}) &= \mathcal{B}_1(\wedge^m) \otimes \mathcal{B}_0(\wedge_{m+1}) \\ &\quad + \mathcal{B}_0(\wedge^m) \otimes (\mathbb{1} - \mathcal{B}_0(\wedge_{m+1})). \end{aligned} \tag{15}$$

Because of the symmetry the recursive relationship above is the same no matter which single copy is used for recurrence. In the following we shall prove via induction

$$\mathcal{D}_1(\wedge^m) = \left(1 + \frac{m}{3}\right) \left(\frac{3}{4}\right)^m. \tag{16}$$

At first we notice the above LHV bound is attained when all the variables are assigned to value +1, i.e., $b_i = \langle \mathcal{B}_0(\wedge_i) \rangle_c = 3/4$ for all $i \leq m + 1$, so that we have only to prove that it is the upper bound of all LHV values. Suppose that Eq. (16) holds true for m copies. It is easy to check the LHV value $\langle \mathcal{B}_1(\wedge^{m+1}) \rangle_c$ when $b_i = -1/4$ for all i is smaller than the LHV value when $b_i = 3/4$ for all i. Therefore, taking into account of the symmetry, we can suppose without lost of generality $b_{m+1} \geq 0$ so that both b_{m+1} and $(1 - b_{m+1})$ are nonnegative. By noticing $\mathcal{D}_0(\wedge^m) = (\frac{3}{4})^m \leq \mathcal{D}_1(\wedge^m)$ and $b_{m+1} \leq \frac{3}{4}$ we have

$$\begin{aligned} \langle B_1(\wedge^{m+1}) \rangle_c &\leq (1 - b_{m+1}) \mathcal{D}_0(\wedge^m) + b_{m+1} \mathcal{D}_1(\wedge^m) \\ &\leq \frac{3^m(m+4)}{4^{m+1}}. \end{aligned} \tag{17}$$

Thus we have proved Eq. (16) for $m + 1$. From this LHV bound we see that the violation of the Bell inequalities increases exponentially with the number of qubits.

On the same the graph $\wedge^m$ we consider the case $t = m - 1$ in which the t-coverable set can be easily found to be $\{C | C \cap V_i = \emptyset \text{ for some } i\}$. The corresponding t-error-tolerating Bell operator reads

$$\mathcal{B}_{m-1}(\wedge^m) = \mathbb{1} - \bigotimes_{i=1}^m (\mathbb{1} - \mathcal{B}_0(\wedge_i)), \tag{18}$$

whose LHV value increases with the LHV values $\mathcal{B}_0(\wedge_i)$ for all $i = 1, 2, \ldots, m$. As a result we have the LHV bound

$$\mathcal{D}_{m-1}(\wedge^m) = 1 - \frac{1}{4^m}. \tag{19}$$

The construction above can be summarized as:

Proposition 3: There exists a valid t-error-tolerating Bell inequality that involves only $3(t + 1)$ qubits.

To build a valid Bell inequality, i.e., $\mathcal{D}_t(G) < 1$, it is necessary that $\mathcal{B}_t(G)$ not be the identity, i.e., the t-coverable set $\mathbb{C}_t$ should not be the full set of all the vertex subsets. This condition is necessary for the graph state $|G\rangle$ being a base for some quantum error-correcting code that correct up to $\lfloor \frac{t}{2} \rfloor$-qubit errors [17]. However, Proposition 3 shows that a valid error-tolerating Bell inequality is not necessarily constructed from some $\lfloor \frac{t}{2} \rfloor$-error-correcting codes. This is because by using any copies of 3-qubit star graphs, only 1-error-correcting code can be constructed.

For small n it is possible to do a computer search on all the n-qubit graph states for valid Bell inequalities. Before an exhaustive computer search we notice that firstly for a given Bell operator of form Eq. (10) we can restrict our attention to those LHV models in which all $\mathcal{Z}$-measurements are assigned to value +1. This is because [6] for a given vertex v, if we revert the signs of the LHV values of all $\mathcal{X}_a$ and $\mathcal{Y}_a$ in the neighborhood of v, i.e., $a \in N_v$, and $\mathcal{Z}_v$ and $\mathcal{Y}_v$ then LHV value of every term of the stabilizers $\mathcal{G}_s$ of the graph state remains the same. And the Bell operator is a linear combination of the stabilizers of the graph state. Secondly the LHV bound for isomorphic graph and LC-equivalent graph is the same so that we have to restrict to those non-isomorphic and non-LC-equivalent graphs.

TABLE I: The optimal values $\mathcal{D}_t(n)$ for $3 \le n \le 10$ and $0 \le t \le 2$ with corresponding graphs shown in Fig.1 (alphabetic labels) and explained in the text.

$t \setminus n$	3	4	5	6	7	8	9	10
0	$3/4^a$	3/4	5/8	7/16	$6/16^b$	$10/32^c$	$13/64^d$	$11/64^e$
1	1	1	1	$15/16^\alpha$	$15/16^\alpha$	$29/32^\alpha$	$54/64^\beta$	$48/64^f$
2	1	1	1	1	1	1	$63/64^\beta$	$63/64^\beta$

We denote by $\mathcal{D}_t(n) = \min_{|\mathrm{V}|=\mathrm{n}} \mathcal{D}_t(G)$ the minimal LHV bound, i.e., largest violation, among all graphs on n vertices and its values for $n \le 10$ and $t \le 2$ are documented in Table I with some of the optimal graphs shown in Fig.1. In the case of $t = 0$ the optimal values $\mathcal{D}_0(n)$ for $n = 3, 4, 5, 6$ are attained by the ring graphs as calculated in [6] while our optimal values for $n = 7, 8, 9, 10$, which are attained on the graphs shown in Fig.1b to Fig.1e respectively, improve the corresponding violations in [6]. Interestingly, in the case of $t > 0$, many disconnected graphs made up of complete graphs (or star graphs) can attain the optimal bound. For examples, in Table I, those optimal values labeled with α are attained by the graphs $K_3 \oplus K_{n-3}$ $(n = 6, 7, 8)$ and those values labeled with β by graphs $K_3 \oplus K_3 \oplus K_{n-6}$ $(n = 9, 10)$. In the case of $t \ge 3$ we have always $\mathcal{D}_t(n) = 1$ for $n \le 10$. It should be noted that the graphs attaining some of the optimal values may not be unique and we have only listed one of the optimal graphs.

If we define $\mathcal{N}_t^{op} = \min\{n \big| \mathcal{D}_t(n) < 1\}$ as the smallest number of qubits that are involved in a valid t-error-tolerating Bell inequality then the proposition 3 establishes an upper bound $\mathcal{N}_t^{op} \le 3(t + 1)$. From Table I we see that the upper bound is exact, i.e., the equality sign holds true, in the case of $t = 0, 1, 2$ and it is tempting to conjecture that our upper bound is exact for any t.

In summary we have combined the idea of the Bell inequality via graph states [6] and the idea of the error-correcting Bell inequalities [13] and gained some new sights. First of all a t error-tolerating Bell inequality is not necessarily built on some $\lfloor \frac{t}{2} \rfloor$-error-correcting code and all the graph states are possible candidates for the error-tolerating Bell inequality. Secondly we have established the upper bound $3(t + 1)$ of the minimal number of qubits that are involved in a valid t-error-tolerating Bell inequality and this upper bound is exact for $t \le 2$ as a result of an exhaustive computer search. It is noteworthy that we have reduced the number of qubits from 11 to 6 that is involved in a 1-error-tolerating Bell inequality. Therefore an experimental test is feasible [20, 21]. Finally because the stabilizer states are LC-equivalent to the graph states [15] our results hold in fact for all the stabilizer states.

We acknowledge the financial support of NNSF of China (Grants No. 10705025, and No. 10675107) and the A*STAR Grant No. R-144-000-189-305.

[1] J.S. Bell, Physics (Long Island City, N.Y.) **1**, 195 (1964).

[2] P. Pearle, Phys. Rev. D **2**, 1418 (1970); E. Santos, Phys. Rev. A **46**, 3646 (1992).

[3] L.E. Ballentine, Am. J. Phys. **55**, 785 (1987); A. Peres, Found. Phys. **29**, 589 (1999); L. Vaidman, Phys. Lett. A **286**, 241 (2001).

[4] D. M. Greenberger, M. A. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Theory, and Conceptions of the Universe*, edited by M. Kafatos (Kluwer Academic, Dordrecht, Holland, 1989), p. 69.

[5] N.D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990); M. Ardehali, Phys. Rev. A **46**, 5375 (1992).

[6] O. Gühne, G. Tóth, P. Hyllus and H.J. Briegel, Phys. Rev. Lett. **95**, 120405 (2005).

[7] A. Cabello, Phys. Rev. Lett. **95**, 210401 (2005); A. Cabello, O. Gühne, and D. Rodríguez, Phys. Rev. A **77**, 062106 (2008).

[8] M. Hein, J. Eisert, and H.J. Briegel, Phys. Rev. A **69**, 062311 (2004).

[9] R. Raussendorf and H.J. Briegel, Phys. Rev. Lett. **86**, 5188 (2001).

[10] D.P. DiVincenzo and A. Peres, Phys. Rev. A **55**, 4089 (1997).

[11] V. Scarani, A. Acín, E. Schenck, and M. Aspelmeyer, Phys. Rev. A **71**, 042325 (2005); G. Tóth, O. Gühne, and H.J. Briegel, *ibid.* **73**, 022303 (2006); L. -Y. Hsu, *ibid.*, **73** 042308 (2006);

[12] J. Barrett, C. M. Caves, B. Eastin, M. B. Elliott, and S. Pironio, Phys. Rev. A **75**, 012103 (2007); C. Wu, Y. Yeo, L. C. Kwek, and C. H. Oh, *ibid.* **75**, 032332 (2007); O. Gühne and A. Cabello, *ibid.* **77**, 032108 (2008).

[13] T.A. Walker, F.A.C. Polack and S.L. Braunstein, Phys. Rev. Lett. **101**, 080501 (2008).

[14] D. Schlingemann and R. F. Werner, Phys. Rev. A **65**, 012308 (2001).

[15] D. Schlingemann, Quant. Inf. Comp. **2**, 307 (2002); M. Grassl, A. Klappenecker, and M. Rotteler, IEEE Int. Symp. Inform. Theory Proceedings, pp 45 (2002).

[16] S. Yu, Q. Chen, C.H. Lai, and C.H. Oh, Phys. Rev. Lett. **101**, 090501 (2008).

[17] S. Yu, Q. Chen, and C.H. Oh, e-print arXiv: 0709.1780.

[18] D. Hu, W. Tang, M. Zhao, Q. Chen, S. Yu, and C. H. Oh, Phys. Rev. A **78**, 012306 (2008); S. Y. Looi, L. Yu, V. Gheorghiu, and R.B. Griffiths, *ibid.* **78**, 042303 (2008).

[19] A. Cross, G. Smith, J. Smolin, and B. Zeng, IEEE Trans. Inf. Theory **55**, 433 (2009).

[20] P. Walther, M. Aspelmeyer, K. J. Resch, and A. Zeilinger, Phys. Rev. Lett. **95**, 020403 (2005); C.-Y. Lu, X.-Q. Zhou, O. Gühne, W.-B. Gao, J. Zhang, Z.-S. Yuan, A. Goebel, T. Yang, and J.-W. Pan, Nat. Phys. **3**, 91 (2007).

[21] W.-B. Gao, C.-Y. Lu, X.-C. Yao, P. Xu, O. Gühne, A. Goebel, Y.-A. Chen, C.-Z. Peng, Z.-B. Chen, and J.-W. Pan, e-print arXiv:0809.4277.